JOURNAL OF APPROXIMATION THEORY 26, 212–218 (1979)

The Lipschitz Continuity of the Metric Projection

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Communicated by John R. Rice

Received December 1, 1977

We investigate the Lipschitz continuity of the best approximation operator from a Hilbert (Banach) space into an approximatively compact subset. We study the notion of directional radius of curvature and show how the Lipschitz continuity of the metric projection depends on it.

1. INTRODUCTION

Let M be a set in a Hilbert space H. We define the metric projection P by $P(x) = \{y \mid y \in M \text{ and } \inf_{m\in M} \| x - m \| = \| x - y \|\}$ where x is any element of H.

In our discussion we shall assume that M is an approximatively compact set which insures that $P(x)$ is not empty.

If M is a closed convex set then it is well known that each $P(x)$ is a singleton and $\| P(y) - P(x)\| \leqslant \| y - x \|$. When M is a C^2-approximatively compact manifold, Jerry Wolfe in [6] has proved that P is a singleton, differentiable in an open dense subset A ($A \cap M = \varnothing$) of H and the present author has shown in [1] that $\| P'(x)\| = \rho/(\rho - r)$ in A where $r = \| x - P(x)\|$ and ρ is the radius of curvature of M at $P(x)$ in the direction $x - P(x)$.

If M is a closed subset in R^n, Federer in [3] has proved that under suitable conditions, P is a singleton and $\| P(y) - P(x)\| \leqslant q[/(q - r)] \| x - y \|$, where $q \leqslant \omega(P(y))$ and ω is the reach; see definition 11–13 in [4].

In this paper we give best estimates for

$$\overline{\lim_{y\to x}} \frac{\| P(y) - P(x)\|}{\| y - x \|}$$

in various settings.

2. DEFINITIONS

A subset M of a Banach space B is called approximatively compact if, for each $x \in B$ and each sequence $\{m_n\} \subset M$ such that $\| x - m_n \| \to \inf_{m\in M} \| x - m \|$, there exists a subsequence $\{m_{n_k}\}$ converging to a point of M.

0021-9045/79/070212-07$02.00/0

3. Radius of Curvature

Let x be in a Banach space B and m be a closest point to x from $M(x)$. Also let $v = (x - m)/\| x - m \|$.

We consider the line $m + tv$, $t \in R$, and points $\mu \in M$ close to m such that

$$| t | = \| m + tv - \mu \|$$

for some t, $-\infty < t < \infty$. If the above equation holds for no finite t, set $t = \infty$. We now define the directional radius of curvature of M at m in the direction v, $\rho(m, v)$, by

$$\rho(m, v) = \lim_{\epsilon \downarrow 0} t_\epsilon \text{ where } t_\epsilon = \left[\sup_{0 < \|\mu - m\| < \epsilon} \left\{ \frac{1}{t} \,\Big|\, | t | = \| m - \mu + tv \| \right\} \right]^{-1},$$

It can be shown, see [1], that if M is a C^2 manifold in a Hilbert space H and if M is locally represented by f around m, then

$$\frac{1}{\rho(m, v)} = \max_{\|w\|=1} \frac{\langle Aw, w \rangle}{\langle Bw, w \rangle}$$

where

$$m = f(a),\ B = f'(a)^T f'(a),\ A = \left(\left\langle v, \frac{\partial^2 f(a)}{\partial t_i\, \partial t_j} \right\rangle \right) .$$

The lemma below, which is a generalization of Theorem 11–8 in [4], illustrates the geometric significance of $\rho(m, v)$. For convex sets M, $\rho(m, v) \leqslant 0$.

Lemma 3.1. *Let M be a closed set in a Banach space B, $x \in B$ and $m \in P(x)$, $m \neq x$. Let $0 < \delta < | \rho(m, v)|$, and set $y = m + (\rho(m, v) - \delta)v$. Then there exists $\epsilon_\delta > 0$ and an open ball $B(m, \epsilon_\delta)$ such that:*

(a) *If $\rho(m, v) > 0$ and $m' \in M \cap B(m, \epsilon_\delta)$, then $\| y - m \| \leqslant \| y - m' \|$.*

(b) *If $\rho(m, v) < 0$, then $\| y - m' \| \leqslant \| y - m \|$ for all m' in $M \cap B(m, \epsilon_\delta)$.*

Proof. (a) Suppose not. Then for $n = 1, 2, \ldots$ there is an $m_n \neq m$ in $M \cap B(m, 1/n)$ such that

$$\| y - m \| > \| y - m_n \|.$$

Choose $a_n > 0$, so that $\| y - a_n v - m \| = \| y - a_n v - m_n \|$. Substituting for y and setting $\rho = \rho(m, v)$, we get

$$| \rho - \delta - a_n | = \| m - m_n + (\rho - \delta - a_n)v \|.$$

Then $\overline{\lim}_{n\to\infty} 1/(\rho - \delta - a_n) \geqslant 1/(\rho - \delta) > 1/\rho$, contradicting the definition of ρ.

(b) Suppose not. Then, for $n = 1, 2,\ldots$, there is an $m_n \neq m$ in $M \cap B(m, 1/n)$ such that $\| y - m \| < \| y - m_n \|$. By definition, $\rho = \rho(m, v) \leqslant \underline{\lim}_{n\to\infty} t_n < 0$ where

$$| t_n | = \| m - m_n' + t_n v \| .$$

Then clearly $\rho - \delta \leqslant t_n \leqslant 0$ for n sufficiently large and $\| y - m \| = | \rho - \delta | = | \rho - \delta - t_n | + | t_n | \geqslant \|(\rho - \delta - t_n)v \| + \|(m - m_n) + t_n v \| \geqslant \|(m - m_n) + (\rho - \delta)v \| = \| y - m_n \|$, a contradiction.

4. The Pointwise Lipschitz Continuity of the Metric Projection

We shall need the following lemma [5, p. 388].

Lemma 4.1. *Let M be an approximatively compact subset of a Banach space B. Suppose $x \in B$ has m as a unique closest point from M and let $\{x_k\}$ be a sequence converging to x and $\{m_k\}$ a corresponding sequence of closest points in M. Then $m_k \to m$.*

The following theorem establishes the pointwise Lipschitz continuity of the metric projection in terms of the radius of curvature.

Theorem 4.1. *Let M be an approximatively compact set in a Hilbert space H. Suppose $x \in H$, $x \notin M$, has m as a unique closest point from M. Assume $\rho(m, x - m/\| x - m \|) = \rho \neq \| x - m \| = r$. Then if $m_y \in P(y)$, we have*

$$\overline{\lim_{y\to x}} \frac{\| m_y - m \|}{\| y - x \|} \leqslant \frac{2\rho}{\rho - r} .$$

Proof. Without loss of generality assume $m = 0$ and $\rho > 0$. (The case $\rho < 0$ can be treated similarly and the case $\rho = 0$ can be reduced to that of $\rho > 0$). Since $m = 0, v = (x - m)/\| x - m \| = x/r$. By Lemmas 3.1 and 4.1, for any $\delta > 0$, $0 < \delta < \rho$, if y is sufficiently close to x, we have

$$| \rho - \delta | \leqslant \|(\rho - \delta)v - m_y \|,$$

and so

$$(\rho - \delta)^2 \leqslant (\rho - \delta)^2 - 2\,\frac{\rho - \delta}{r} \langle x, m_y\rangle + \| m_y \|^2.$$

Hence

$$2\,\frac{\rho - \delta}{r} \langle x, m_y\rangle \leqslant \| m_y \|^2. \tag{1}$$

It is also clear that $\| y - m_y \|^2 \leqslant \| y \|^2$, from which we obtain

$$\| m_y \|^2 \leqslant 2\langle y, m_y \rangle. \tag{2}$$

Write (2) as $\| m_y \|^2 \leqslant 2\langle y - x, m_y \rangle + 2\langle x, m_y \rangle$. We substitute (1) into this to obtain

$$\| m_y \|^2 \leqslant 2\langle y - x, m_y \rangle + \frac{r}{\rho - \delta} \| m_y \|^2,$$

$$\frac{\rho - \delta - r}{\rho - \delta} \| m_y \|^2 \leqslant 2\langle y - x, m_y \rangle. \tag{3}$$

By Lemma 3.2 in [2], $r < \rho$, so that if δ is sufficiently small, then $\rho - \delta - r > 0$. Now inequality (3) implies that

$$\| m_y \|^2 \leqslant \frac{2(\rho - \delta)}{\rho - \delta - r} \| y - x \| \| m_y \| \tag{6}$$

from which we get

$$\overline{\lim_{y \to x}} \frac{\| m_y \|}{\| y - x \|} \leqslant 2 \frac{\rho - \delta}{\rho - \delta - r},$$

and hence

$$\overline{\lim_{y \to x}} \frac{\| m_y \|}{\| y - x \|} \leqslant \frac{2\rho}{\rho - r}.$$

EXAMPLE. Let H be Euclidean 2-space and $M = \{e^{i\theta} \mid \pi \leqslant \theta \leqslant 2\pi\} \cup \{e^{i\theta_n} \mid n = 1, 2, 3,...\}$, where $0 < \cdots < \theta_{n+1} < \theta_n < \cdots < \theta_1 < \pi$, $\theta_n \to 0$ and $\theta_{n+1}/\theta_n \to 0$. Then M is approximatively compact. Let $x = 1 - r$ with $0 < r < 1$. It is clear that 1 is the closest point to x in M. Also $\rho(1, -1) = 1$. Consider

$$y_n = (1 - r) \cos \left(\frac{\theta_n + \theta_{n+1}}{2}\right) e^{i} \left(\frac{\theta_n + \theta_{n+1}}{2}\right), n = 1, 2,....$$

Then $\rho e^{i\theta_n}$ and $\rho e^{i\theta_{n+1}}$. are the closest points in M to y_n. Computation gives

$$| x - y_n | = (1 - r) \sin \left(\frac{\theta_n + \theta_{n+1}}{2}\right) \text{ and } | 1 - e^{i\theta_n} | = 2 \sin \frac{\theta_n}{2},$$

so that

$$\lim_{n \to \infty} \frac{| 1 - e^{i\theta_n} |}{| x - y_n |} = \lim_{n \to \infty} \frac{2 \sin (\theta_n/2)}{(1 - r) \sin ((\theta_n + \theta_{n+1})/2)} = \frac{2}{1 - r}.$$

This example shows the sharpness of Theorem 4.1. We now obtain a related result for closed convex sets which is sharper. It should be remarked that for such sets $\rho \leqslant 0$.

THEOREM 4.2. *Let M be a closed convex set in a Hilbert space H. Then for $x \notin M$,*

$$\overline{\lim_{y \to x}} \frac{\| P(y) - P(x) \|}{\| y - x \|} \leqslant \frac{2\rho}{2\rho - r}$$

where $\rho = \rho(P(x), (x - P(x))/r)$ and $r = \| x - P(x) \|$.

Proof. We may assume $P(x) = 0$. It follows from Lemma 3.1(b) that for every $\delta > 0$ there exists $\epsilon(\delta) > 0$ such that if $\| P(y) \| < \epsilon(\delta)$, then

$$\left\| (\rho - \delta) \frac{x}{r} - P(y) \right\|^2 \leqslant \left\| (\rho - \delta) \frac{x}{r} \right\|^2$$

from which we get

$$\langle x, P(y) \rangle \leqslant \frac{r}{2(\rho - \delta)} \| P(y) \|^2. \tag{1}$$

Also by convexity of M we have $\langle y - P(y), -P(y) \rangle \leqslant 0$ which implies

$$\| P(y) \|^2 \leqslant \langle y, P(y) \rangle = \langle y - x, P(y) \rangle + \langle x, P(y) \rangle. \tag{2}$$

From (1) and (2), we obtain, using Schwarz's inequality,

$$\frac{\| P(y) \|}{\| y - x \|} \leqslant \frac{2(\rho - \delta)}{2(\rho - \delta) - r}.$$

Hence

$$\overline{\lim_{y \to x}} \frac{\| P(y) \|}{\| y - x \|} \leqslant \frac{2(\rho - \delta)}{2(\rho - \delta) - r}$$

and therefore

$$\overline{\lim_{y \to x}} \frac{\| P(y) \|}{\| y - x \|} \leqslant \frac{2\rho}{2\rho - r}.$$

EXAMPLE. Let H be as in the previous example and let M be the convex set whose boundary is the union of the semicircle $e^{i\theta}$, $\pi \leqslant \theta \leqslant 2\pi$ and the polygon whose vertices are $e^{i\theta_n}$ where $0 \leqslant \cdots < \theta_n < \cdots < \theta_2 < \theta_1 < \pi$, $\theta_n \to 0$ and $\theta_{n+1}/\theta_n \to 0$. Let $v_n = e^{i\theta_n}$, $n = 1, 2,...$; $x = 1 + r$. Let $r > 0$, $x = (1 + r, 0)$. Set

$$y_n = \frac{\langle x - v_n, v_n + v_{n+1} \rangle}{\| v_n + v_{n+1} \|^2} (v_n + v_{n+1}) + v_n, \; n = 1, 2,....$$

Then, for $n = 1, 2,...$, $P(y_n) = v_n$. It can be shown that

$$\lim_{n \to \infty} \frac{\| P(y_n) - P(x) \|}{\| y_n - x \|} = \frac{2}{2 + r}.$$

Observe that here $\rho(P(x), x - P(x)) = -1$. We now extend Theorem 4.1 to n-dimensional C^1 manifolds in Banach spaces.

We make the following assumptions about the C^1 manifold M lying in a Banach space B.

(1) M can be represented locally by a C^1 function f.

(2) f is a relatively open map in its domain of definition.

(3) $f'(a)(R^n)$ is an n-dimensional subspace of B.

(4) Letting $m = f(a)$ we define the tangent plane of M at m to be $T_m \equiv m + f'(a)(R^n)$.

THEOREM 4.3. *Let M be a C^1 approximatively compact n-dimensional manifold in a Banach space B whose norm is uniformly C^2 except at zero. Assume $x \in B$, $x \notin M$ and that P is a singleton in a B neighborhood of x (and hence continuous at x). Assume also that $\nabla^2 \| x - P(x)\|$ is positive definite on the tangent space of M at $P(x)$ and $\rho = \rho(P(x), (x - P(x))/r) \neq \| x - P(x)\| = r$. Then*

$$\overline{\lim_{y \to x}} \frac{\| P(y) - P(x)\|}{\| y - x \|} \leqslant C(x) \frac{\rho}{\rho - r}$$

where $C(x)$ is a positive constant depending on x.

Proof. By a translation we can assume $P(x) = 0$. Let $\rho = \rho(0, x/r)$ and assume $\rho > 0$ (the proof for $\rho \leqslant 0$ is similar). By Lemma 3.1, for any δ, $0 < \delta < \rho$, and for y sufficiently close to x, we have

$$\rho - \delta \leqslant \left\| (\rho - \delta) \frac{x}{r} - P(y) \right\|. \tag{1}$$

Also, trivially,

$$\| y - P(y)\| \leqslant \| y \|. \tag{2}$$

We use the first 3 terms of the Taylor expansion of the norm in (1) to obtain

$$\rho - \delta \leqslant (\rho - \delta) - \nabla \| x \|(P(y)) + \frac{1}{2} \frac{r}{\rho - \delta} \nabla^2 \| x \|(P(y))^{(2)} + o(\| P(y)\|^2)$$

and we do the same in (2) to obtain

$$\| y \| - \nabla \| y \| (P(y)) + \tfrac{1}{2}\nabla^2 \| y \| (P(y))^{(2)} + o(\| P(y)\|^2) \leqslant \| y \|. \tag{4}$$

Inequalities (3) and (4) are equivalent, respectively, to

$$\nabla \| x \|(P(y)) \leqslant \frac{1}{2} \frac{r}{\rho - \delta} \nabla^2 \| x \|(P(y))^{(2)} + o(\| P(y)\|^2) \tag{5}$$

and

$$\tfrac{1}{2}\nabla^2 \| y \|(P(y))^{(2)} \leqslant \nabla \| y \|(P(y)) + o(\| P(y)\|^2). \tag{6}$$

Combining (5) and (6) we get

$$\frac{1}{2}\nabla^2 \| y \|(P(y))^{(2)} \leqslant (\nabla \| y \| - \nabla \| x \|)(P(y)) + \frac{1}{2}\frac{r}{\rho - \delta}\nabla^2 \| x \|(P(y))^{(2)} + o(\| P(y)\|^2)$$

from which we obtain

$$\frac{1}{2}\left(1 - \frac{r}{\rho - \delta}\right)\nabla^2 \| x \|(P(y))^{(2)} \leqslant (\nabla \| y \| - \nabla \| x \|)(P(y)) + o(\| P(y)\|^2) \tag{7}$$

because as $y \to x$, $P(y) \to 0$ and $(\nabla^2 \| y \| - \nabla^2 \| x \|)(P(y))^{(2)} = o(\| P(y)\|^2)$. Now $\underline{\lim}_{y\to x}[\nabla^2 \| x \| (P(y))^{(2)}]/\| P(y)\|^2 \geqslant C > 0$ by our hypothesis on ∇^2. Also $|(\nabla \| y \| - \nabla \| x \|)\, P(y)| \leqslant k \| y - x \| \| P(y)\|$ where

$$k = \sup_{0\leqslant t\leqslant 1} \| \nabla^2 \| tx + (1 - t)\, y \| \| .$$

So by (7)

$$\overline{\lim_{y\to x}} \frac{\| P(y)\|}{\| y - x \|} \leqslant k' \frac{\rho - \delta}{\rho - \delta - r}$$

where $k' > 0$. Hence

$$\overline{\lim_{y\to x}} \frac{\| P(y)\|}{\| y - x \|} \leqslant k' \frac{\rho}{\rho - r} .$$

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